Universiteit Leiden
Wiskunde en Natuurwetenschappen

Toric geometry

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As a $\sim\#$ theorist, I always thought toric geometry was kind of stupid - all the varieties are birat. to $\mathbb{A}^n$, so we understand distribution of rat. pts etc!

Some people work on varieties that are only 'geometrically toric', ~~both~~ (DP etc), but then generally they don't use toric methods, so again no pt learning TG.

However, ~3 mths ago this changed, as I realised TG was exactly providing <u>local</u> models of certain complicated constructions on $\overline{M}_{g,n}$ (see later weeks...).

The keyword is <u>local</u>, esp. in etale top. Then any (relative...) NCD gives rise to a local toric structure.

One nice thing that log geom. does is to give a machine for working w. et-locally toric things. (Toroidal geom does so too, but in less flexible - eg. only works in 'relative NCD' setting, & ~~...~~ eg. no fun toroidal str. on a pt!).

So today I will do some toric geom., & you should bear in mind that log geom. allows us to use this as a 'local model' for a huge range of situations (eg. whenever you have an NCD).

Ref: Fulton, Toric ~~Geometry~~ Varieties.

We work relative to a ring R. ~~regular~~

Notation

~~N is a monoid, $\simeq \mathbb{Z}^n$ gp for some $n \geq 0$.~~

~~$N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$~~

~~$M = \mathrm{Hom}(N, \mathbb{Z})$~~

? do we want this?!

Lattice = a free $\mathbb{Z}$-module of fin. rk. (+ symmetric bilinear form)

N = lattice, $\simeq \mathbb{Z}^n$ some n, + usual inner product.

$M = \mathrm{Hom}_{gp}(N, \mathbb{Z})$, with dual inner product $\langle -, - \rangle$.

$N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R} \simeq \mathbb{R}^n$.

Cone (convex polyhedral): a set of the form

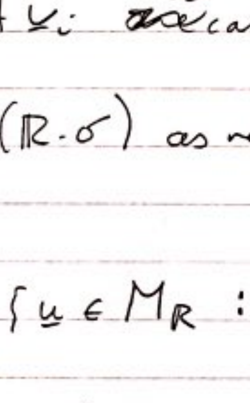
$\sigma = \{ r_1 \underline{v}_1 + \dots + r_s \underline{v}_s \in N_\mathbb{R} : r_i \geq 0 \}$ some $\underline{v}_1, \dots, \underline{v}_s \in N_\mathbb{R}$

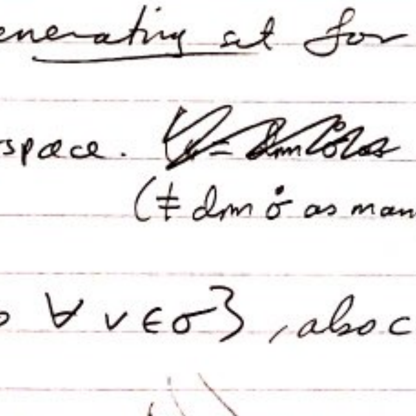
Such a set of $\underline{v}_i$ is called a generating set for σ.

$\dim \sigma = \dim (\mathbb{R} \cdot \sigma)$ as real vectorspace. ~~[illegible]~~
($\neq \dim \mathring{\sigma}$ as manifold...)

Dual: $\sigma^\vee = \{ \underline{u} \in M_\mathbb{R} : \underbrace{\langle \underline{u}, \underline{v} \rangle}_{= \underline{u}(\underline{v})} \geq 0 \ \forall v \in \sigma \}$, also c.p. cone.

eg $n = 2$, $N = \mathbb{Z}^2 = \Pi$.





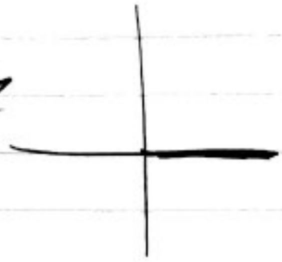
Facts: • If $v_0 \notin \sigma$ then $\exists\, u_0 \in \sigma^\vee$ s.t. $\langle u_0, v_0 \rangle < 0$.

• $(\sigma^\vee)^\vee = \sigma$.

~~Faces~~

$u \in \sigma^\vee$, then $u^{\perp} = \{v \in \sigma \mid \langle \sigma, v\rangle = 0\}$ 'supporting hyperplane'

<u>Face</u>: $\tau = \sigma \cap u^\perp$, $u \in \sigma^\vee$

eg.  σ faces: σ, $\{0\}$.

 faces: 4 of them.

<u>Facts</u>: · faces are also c.p. cones.
· intersection of faces is face.
· face of a face is a face.
[facet := face of codim 1.
· every proper face is $\subseteq$ some facet].

· Topological boundary of σ = union of proper faces.

<u>Lem</u> (Gordan): $S_\sigma := \sigma^\vee \cap M$ is a finitely generated monoid.

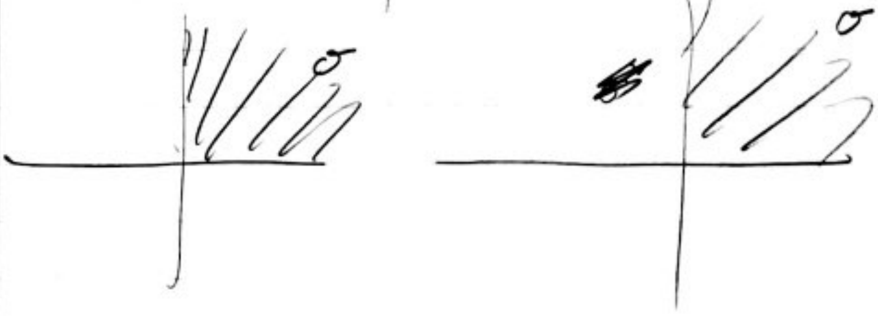
This clearly needs the cone to be rational, i.e. generated by rational vectors. Presumably we should assume this from now on?!

<u>Def</u> σ is <u>strongly convex</u> if it contains no nonzero linear subspace / line.
($\Leftrightarrow$ $\mathbb{R}\cdot\sigma^\vee = M_{\mathbb{R}}$)

<u>Def</u> / The affine toric variety assoc to σ is
let σ str. conv. p. cone. $U_\sigma := \mathrm{Spec}\, R[S_\sigma]$ (Gordan $\Rightarrow$ fin. type).

$= \sigma^\vee \cap M$.

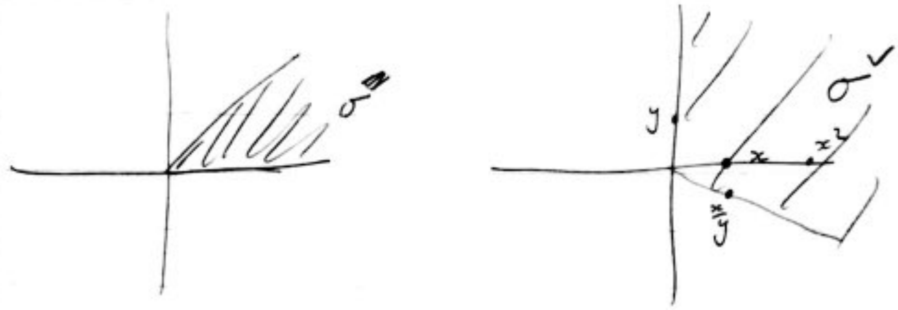
eg. $N = \mathbb{Z}^2 = M$. ~~$\sigma^\vee \cap M =$~~



$\sigma^\vee \cap M \simeq \mathbb{N}^2$,

so $\operatorname{Spec} R[S_\sigma] \simeq \mathbb{A}^2_R$.
U_σ

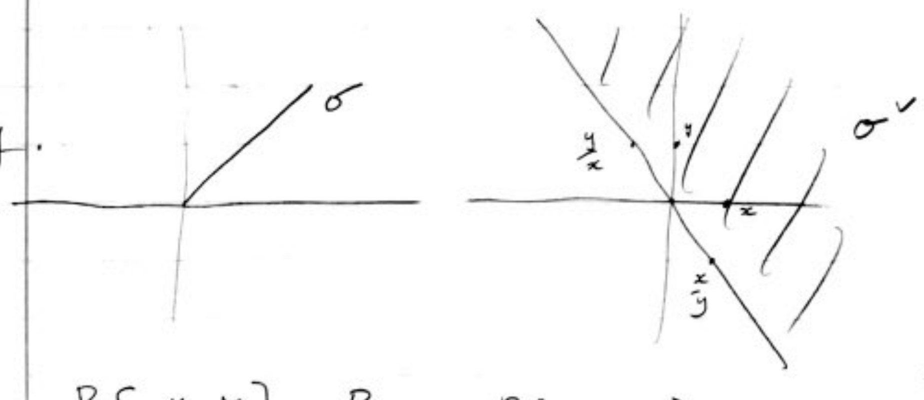
eg.



$$R[\sigma^\vee \cap M] = R\frac{[x, y, t]}{(yt - x)} \quad (\cong R[y, t])$$

(affine patch of a blowup).

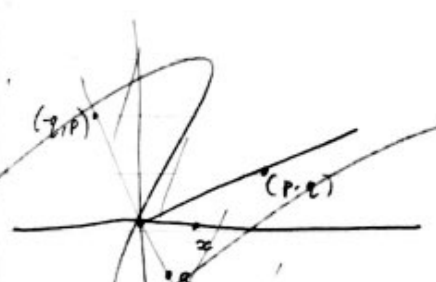
eg.



$$R[\sigma^\vee \cap M] = R \qquad \frac{R[x, a, a^{-1}]}{(ay - x)}$$

universal for $\frac{x}{y}$ being a unit.

(cf. beige notebook...)



Say

~~$R[x, a, a^{-1}]$~~

Lem If τ a face of σ then get canonical $U_\tau \hookrightarrow U_\sigma$ (5)

Idea: have $\tau \to \sigma$, $\rightsquigarrow \sigma^\vee \to \tau^\vee$, $\rightsquigarrow \sigma^\vee \cap M \to \tau^\vee \cap M$, $\rightsquigarrow R[\sigma^\vee \cap M] \to R[\tau^\vee \cap M]$

$\rightsquigarrow U_\tau \longrightarrow U_\sigma$

Toric varieties

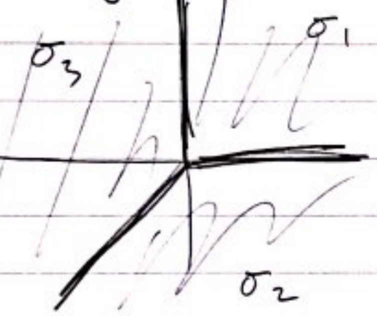
Def A fan Δ in N is a set of s.c.p. cones $\sigma \subseteq N_{\mathbb{R}}$ s.t.

- Δ closed under taking faces.
- if $\sigma, \sigma' \in \Delta$ then $\sigma \cap \sigma'$ is a face of σ (& of σ').

Given fan Δ, define toric variety $X(\Delta)$ by taking union of $U_\sigma : \sigma \in \Delta$, & glueing U_σ to U_τ along $U_{\sigma \cap \tau}$ via above lemma.

logstructures glue from affine parts.

eg

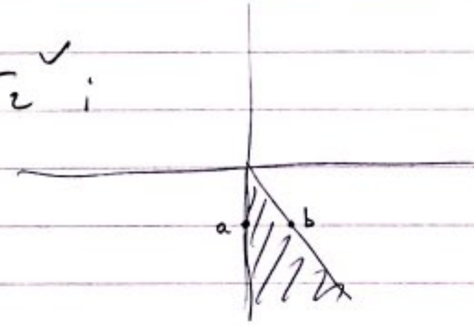


$U_{\sigma_1} = \operatorname{Spec} R[x,y]$

~~$U_{\sigma_2} = R[$~~

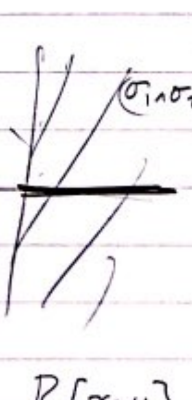
$U_{\sigma_2} = \operatorname{Spec} R[a,b]$

$\sigma_2^\vee$:



" $a = \frac{1}{y}$, $b = \frac{x}{y}$ "

$\sigma_1 \cap \sigma_2$:



$U_{\sigma_1 \cap \sigma_2} = \operatorname{Spec} R[x, c, c^{-1}]$

" $c = y$ "

Glueing: $R[x,y] \to R[x, c, c^{-1}]$ ... just get $\mathbb{P}^2_{\mathbb{R}}$!

$y \longmapsto c$

$R[a,b] \to$

$a \longmapsto c^{-1}$

$b \longmapsto x c^{-1}$

Why is it nice to know a variety is toric?

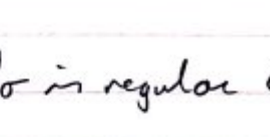
(This is just one random application, but it's fun!)
Toric varieties are always normal & separated, but may or may not be regular.

Let's work over a perfect field (I'm sure not needed, but I have not checked ...).

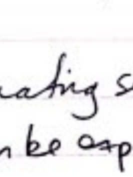
Reduce to affine case, so σ s.c.p. cone, $U_\sigma = \operatorname{Spec} k[\sigma^\vee \cap M]$.

<u>Prop</u>. U_σ is regular $\Leftrightarrow$ $\exists$ a generating set G for σ s.t. G can be expanded to a <u>basis</u> for N as $\mathbb{Z}$-module.

<u>Eg</u> if ~~d~~ dim $\sigma = 0$ or 1, regular.

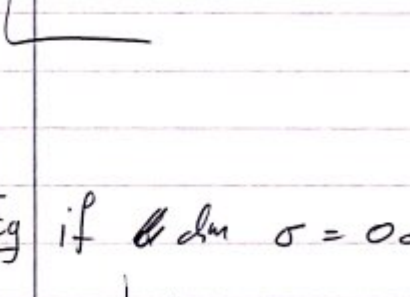


regular



regular.

eg.



& $\det \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \neq 1$

~~Prop~~. Say σ <u>simplicial</u> ~~iff~~ if it can be gen. by $(\dim \sigma)$ elts.

So regular $\Rightarrow$ simplicial.
Say σ simplicial, let $v_1, \dots, v_k$ be first gens along edges,
~~Prop~~ multiplicity $\operatorname{mult}(\sigma) := [N_\sigma : \mathbb{Z}v_1 + \dots + \mathbb{Z}v_k]$.

$N_\sigma := (\mathbb{R}\sigma) \cap N$.

<u>Prop</u> U_σ regular $\Leftrightarrow$ $\operatorname{mult}(\sigma) = 1$.

res. of sings.